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Theory and Methodology

Special cases of the tolerance approach in multiobjective linear programming

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Abstract

In this paper we deal with sensitivity analysis of the Multiobjective Linear Problem. We enlarge the range of meaningful regions of weights that can be handled easily from the tolerance approach point of view. The regions that we propose are important, because they represent the information that the decision maker is able to offer about the relationships between the importance of the different objectives, in terms of marginal substitution rates. Particularly, the well known interval, order and nonhomogeneous order relations are included. We also present an algorithm which reduces the computational effort, in order to get the maximum tolerance percentage. Several examples are included illustrating the results. © 1997 Elsevier Science B.V.

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1. Introduction

Solving a Multiobjective Linear Problem by the weighted sum approach with positive estimated weights, gives an efficient solution to it. The study of this solution sensitivity with respect to the considered weights, is a fundamental issue when looking for the best solution.

Sensitivity analysis is difficult when dealing with perturbations in more than one coefficient or term at a time. The fact that critical regions are always polytopes and the difficulty that the decision maker (DM) finds to deal with such polytopes in practice, makes the focal point of much of the research in sensitivity analysis to find meaningful perturbations of some parameters of the problem, so that the critical regions preserve a certain property.

In this sense, the tolerance approach (Wendell, 1984, 1985; Ravi and Wendell, 1989) presented a new perspective on sensitivity analysis in linear programming, for dealing with simultaneous and independent perturbations of the right hand side terms, of the objective function coefficients, and of the matrix coefficients. This approach incorporates the possibility of using a priori information about the variability of the coefficients, in order to obtain larger tolerance percentages.

Hansen et al. (1989) proposed the tolerance approach to address sensitivity analysis of the Multiobjective Linear Problem. They show how to calculate

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a maximum tolerance percentage by which weights may deviate simultaneously and independently from their estimated values, yielding the same optimal basic solution. Specifically they address two useful particular cases: when there is no information about the variability and when the weights are known to vary within intervals. However there are some other interesting cases that may be addressed easily by this general approach.

Usually the decision maker does not know precisely the values of the objective function weights, but he may be able to specify some linear relations that the weights have to verify. For instance, it may be easy for him to give an order on the importance of the objectives. Thus if the objectives were ordered in increasing order of preference the weights would verify

$$0 \le \lambda_1 \le \lambda_2 \le \dots \le \lambda_p.$$

More generally, the decision maker may be able to establish comparisons between the importance of an objective and a linear combination of the others.

$$\lambda_i \ge \sum_{j=1}^{p} m_{ij}\lambda_j, \quad \sum_{j=1}^{p} m_{ij} < 1.$$

Notice that this kind of relations express partial information about the minimum marginal substitution rates between the criteria (Carrizosa et al., 1995).

In a general setting we may have the region where the weights can vary as

$$\Lambda = \{\lambda \in \mathbb{R}^p, \quad \lambda \ge 0, \quad \alpha \le M\lambda \le \beta\},$$

where $M \in \mathbb{R}^{p \times p}$.

It means that upper and lower levels on the relations expressing the marginal substitution rates can be established.

It is important to point out that, even without information about weights, they must always be considered nonnegative when dealing with efficient solutions. Otherwise, the approach does not represent sensitivity analysis of the Multiobjective Linear Problem, but sensitivity of a particular linear problem obtained by weighting the individual problems.

This paper focuses on enlarging the range of meaningful regions of weights, that can be handled easily from the tolerance approach point of view. We will center our attention in a particular class of relations, those that generate a nonnegative inverse matrix. Practical conditions for a matrix M having nonnegative inverse, that are interpretable in terms of the information the DM is offering, have been studied recently by Carrizosa et al. (1995). In addition, we will see that the special properties of the information given in this form will enable us to simplify the procedure to calculate the maximum tolerance.

The paper is organized as follows. In Section 2 we state the problem and describe some preliminary results. Section 3 contains the main results. We present some illustrative examples in Section 4. Finally, Section 5 is devoted to the conclusions and possible extensions.

2. Statement of the problem

Consider the multiple objective linear program

$$\begin{array}{ll} \text{Max} & Cx, \\ \text{s.t.} & Ax = b, \\ & x \ge 0. \end{array} \quad (1)$$

where $C \in \mathbb{R}^{p \times n}$, $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$

When solving this problem by the weighted sum approach, each objective $c^r x$ is associated with a positive weight λ_r^0, and all of them are combined into a composite criterion function $\lambda^{0\prime}Cx$. This leads to the following weighted sum problem:

$$\begin{array}{ll} \text{Max} & \lambda^{0\prime}Cx, \\ \text{s.t.} & Ax = b, \\ & x \ge 0. \end{array} \quad (2)$$

Solving (2) with the estimated weights λ_r^0 ($\lambda_r^0 > 0$), one obtains an efficient solution to (1).

We assume that A has full rank, let B denote an optimal basis to (2), K the index set of the nonbasic variables and W the reduced cost matrix associated to basis B in the multiobjective problem of (1). For any matrix L we use the notation L^r to denote the rth column of L. The column of matrix W, w^r, is the reduced costs of the nonbasic variable r, for the different objective functions. Finally, we will denote by u' the transpose of a vector u.

In order to deal with multiplicative perturbations of the estimated weights λ^0, we focus on the following perturbed problem:

$$\begin{aligned} \text{Max} \quad & \sum_{r=1}^{p} (\lambda_r^0 + \gamma_r \lambda_r^0)(c^r x), \\ \text{s.t.} \quad & Ax = b, \\ & x \geq 0, \end{aligned} \tag{3}$$

with $\lambda_r^0 > 0$, $r = 1, \ldots, p$.

It is easy to see that in this formulation γ_r represents the percentage deviation from λ_r^0. Let us denote $\lambda_r = \lambda_r^0 + \gamma_r \lambda_r^0$, so we can write $\lambda = \lambda^0 + D_{\lambda^0}\gamma$, where D_{λ^0} is the diagonal matrix with λ^0 in its diagonal.

For the sake of completeness, we include without proof some results on the maximum tolerance of parameter γ of (3), when the weights are known to vary in a specified polyhedron Λ. Further details can be found in Hansen et al. (1989).

The maximum tolerance, τ^*, is defined as the supremum of the allowable tolerances, being a nonnegative number τ an allowable tolerance if the same basis B is optimal in (3) whenever $\lambda \in \Lambda$ and the absolute value of each perturbation γ_r does not exceed τ. Notice that the maximum tolerance percentage represents the maximum percentage by which weights may deviate simultaneously and independently from their estimated values yielding the same optimal basic solution.

The maximum tolerance is determined by $\tau^* = \min\{\tau_k, k \in K\}$ where τ_k is obtained as follows.

1. $\tau_k = +\infty$ if and only if $\sup\{\lambda' w^k, \lambda \in \Lambda\}$ is nonpositive.
2. If $\tau_k < +\infty$ then $\tau_k = \|\gamma^*\|_\infty$ where γ^* is an optimal solution to $\min\{\|\gamma\|_\infty, \ \lambda \in \Lambda, \ \lambda' w^k = 0\}$ where $\lambda = \lambda^0 + D_{\lambda^0}\gamma$.
3. If $\tau_k < +\infty$ and $\Lambda = \mathbb{R}^p$, then $\tau_k = (-\sum_{r=1}^p w_r^k \lambda_r^0)/(\sum_{r=1}^p |w_r^k| \lambda_r^0)$.

3. Results

In this section, we present results to obtain the maximum tolerance percentage for (3), when the region of variation of weights is given by

$$\Lambda = \{\lambda \in \mathbb{R}^p, \quad \lambda \geq 0, \quad \alpha \leq M\lambda \leq \beta\},$$

where $M \in \mathbb{R}^{p \times p}$, with $M^{-1} \geq 0$ and $M^{-1}\alpha \geq 0$.

It is important to point out that this type of region generalizes both the case of interval weights and of no information, studied in Hansen et al. (1989), once the necessary constraints $\lambda_i \geq 0$ have been imposed.

It should be noted also, that if the DM is only able to supply $k \leq p$ relations, it is possible to extend the information to a $p \times p$ matrix by adding relations $\lambda_i \geq 0$ independent with the former ones.

First of all we give a necessary and sufficient condition to determine whether τ_k is finite or not.

Theorem 1. *If $\Lambda = \{\lambda \in \mathbb{R}^p, \ \lambda \geq 0, \ \alpha \leq M\lambda \leq \beta\}$, with $M^{-1} \geq 0$ and $M^{-1}\alpha \geq 0$, then $\tau_k = +\infty$ if and only if $(w^k)' M^{-1} \hat{y} \leq 0$ where*

$$\hat{y}_r = \begin{cases} \beta_r & \text{if } (w^k)'(M^{-1})^r > 0, \\ \alpha_r & \text{if } (w^k)'(M^{-1})^r < 0, \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$

Proof. Let $\lambda = \lambda^0 + D_{\lambda^0}\gamma$. $\tau_k = +\infty$ if and only if

$$\begin{aligned} \sup \quad & (w^k)'\lambda, \\ \text{s.t.} \quad & \alpha \leq M\lambda \leq \beta, \\ & \lambda \geq 0, \end{aligned}$$

is less than or equal to 0.

By the linear transformation $M\lambda = y$, the problem becomes

$$\begin{aligned} \sup \quad & (w^k)' M^{-1} y, \\ \text{s.t.} \quad & \alpha \leq y \leq \beta, \end{aligned}$$

which has nonnegative optimum value if and only if the stated condition holds. □

The following corollaries are special cases of Theorem 1.

Corollary 1. *If $\Lambda = \{\lambda \in \mathbb{R}^p, \ \lambda \geq 0, \ M\lambda \geq 0\}$ with $M^{-1} \geq 0$ then*

$$\tau_k = +\infty, \quad \textit{if and only if } (w^k)'(M^{-1})^r \leq 0, \quad \forall r = 1, \ldots, p.$$

In a geometric sense, this last condition means that all the generators of the convex cone Λ, are in the same halfspace determined by hyperplane $w^k\lambda = 0$, in which point λ^0 also is.

The cases addressed in Hansen et al. (1989) of weight intervals and of no information can be also obtained as corollaries of Theorem 1.

Corollary 2. *If* $\Lambda = \{\lambda \in \mathbb{R}^p, \quad \alpha \le \lambda \le \beta\}$, *with* $\alpha \ge 0$, *then*

$t_k = +\infty$, *if and only if* $(w^k)'\hat{y} \le 0$,

where

$$\hat{y}_r = \begin{cases} \beta_r & \text{if } w_r^k > 0, \\ \alpha_r & \text{if } w_r^k < 0, \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$

Corollary 3. *If* $\Lambda = \{\lambda \in \mathbb{R}^p, \quad \lambda \ge 0\}$ *then* $\tau = +\infty$ *if and only if* $w^k \le 0$.

Notice that, since we are looking for efficient solutions we have to consider only nonnegative weights. This fact is not considered in Hansen et al. (1989). Nevertheless, when dealing with multiplicative perturbations, the tolerance regions obtained by these authors are included into the nonnegative weights region, what validates their results.

On the other hand, although in general, testing whether or not τ_k is finite involves the resolution of a linear program, in these particular cases it does not. Moreover, the test may be done analyzing matrix W^tM^{-1}. In the special case analyzed in corollary 1, τ_k will be infinite only when every element of column k is nonpositive. Notice also that in any case, τ_k being finite does not depend on the values of λ^0.

Let us assume that the region where weights are known to vary is $\Lambda = \{\lambda \in \mathbb{R}^p, \quad \lambda \ge 0, \quad \alpha \le M\lambda \le \beta\}$, with $M^{-1} \ge 0$, $M^{-1}\alpha \ge 0$.

Theorem 2. *Provided that* τ_k *is finite,* τ_k *is the optimum value of the problem*

$$\begin{aligned} \min \quad & \|D_{1/\lambda^0} M^{-1} y - e\|_\infty, \\ s.t. \quad & w^{kt} M^{-1} y = 0, \\ & \alpha \le y \le \beta. \end{aligned} \qquad \text{(P.k.)}$$

Proof. If $\tau_k < +\infty$ then $\tau_k = \|\gamma^*\|_\infty$ where γ^* is an optimal solution to

$$\min\{\|\gamma\|_\infty, \quad \lambda = \lambda^0 + D_{\lambda^0}\gamma \in \Lambda, \quad \lambda' w^k = 0\},$$

and performing the linear transformation $M\lambda = y$, the result follows. □

Notice that problem (P.k.) becomes the linear problem

$$\begin{aligned} \min \quad & z, \\ \text{s.t.} \quad & M^{-1}y - z\lambda^0 \le \lambda^0, \\ & M^{-1}y + z\lambda^0 \ge \lambda^0, \\ & (w^k)' M^{-1} y = 0, \\ & \alpha \le y \le \beta, \end{aligned} \qquad \text{(LP.k.)}$$

that has only $2p+1$ constraints, $p+1$ variables and $2p$ slack variables, while in the general case it should have $3p+1$ constraints, $p+1$ variables and $3p$ slack variables.

Based on this transformation, the following algorithm enables us to compute the maximum tolerance percentage, solving a minimum number of linear problems.

1. Let $J = \{k \in K, \quad \tau_k < +\infty\}$, $C = \{\emptyset\}$, $\tau = +\infty$; For each $k \in J$ do $t_k = (-\sum_{r=1}^p w_r^k \lambda_r^0)/(\sum_{r=1}^p |w_r^k| \lambda_r^0)$;
2. Let $t_h = \min\{t_k, \quad k \in J\}$; Let $\lambda_r^h = \lambda_r^0 + \gamma_r^h \lambda_r^0$, where $\gamma_r^h = t_h sg(w_r^h)$; If $\lambda^h \in \Lambda$, then $\tau_h = t_h$, $C = C \cup \{h\}$, $\tau^* = \tau_h$; otherwise proceed to step 3.
3. Compute τ_h as the solution to (P.k); Let $C = C \cup \{h\}$; if $\tau_h < \tau$, then $\tau = \tau_h$, $J = \{k \in J, \quad t_k < \tau\} \setminus \{h\}$; otherwise $J = J \setminus \{h\}$.
4. If $J = \{\emptyset\}$ then $\tau^* = \tau$; otherwise return to step 2.

The above algorithm works as follows: for those indices whose τ_k is finite, we compute an initial solution using the formula for the unconstrained case. For the index h where the minimum is attained, we check if the perturbed weights generated are included in the region Λ, in this case, the initial value is the maximum tolerance. Observe that the other τ_k can only increase or remain equal when introducing additional information. Otherwise, we recompute τ_h as the solution to (P.k), and consider all the indices whose present values are strictly less than the computed value for τ_h. If there are not such indices then we have obtained the maximum tolerance percentage. Otherwise, we repeat the procedure.

We also keep register of the indices whose real τ_h has been computed.

Once the maximum tolerance percentage has been obtained as a measure of the sensitivity of an efficient solution to changes in the weighting coefficients, the decision maker may want to explore the impact on sensitivity of different assumptions about a priori information available. The following results can be used in this sense.

Assume that τ^* is computed for $\Lambda = \{\lambda \geq 0, M\lambda \geq 0\}$, $M^{-1} \geq 0$, and we want to analyze the effect on the tolerance percentage, of the addition of a new constraint, $a\lambda \geq 0$, on the weighting coefficients.

Let us denote $\Lambda_a = \{\lambda \geq 0, M\lambda \geq 0, a\lambda \geq 0\}$ and $v = aM^{-1}$, then it is easy to check the following facts on the relationships between Λ and Λ_a.

1. $v \geq 0 \Leftrightarrow \Lambda_a = \Lambda$ (the new constraint is redundant).
2. $v \leq 0 \Leftrightarrow \Lambda_a = \emptyset$ (the system is inconsistent).
3. Otherwise $\Lambda_a \subset \Lambda$ (The weight region is reduced).

Based on these assertions, we analyze some particular useful cases. A proof of the following proposition can be found in Mármol et al. (1996).

Proposition 1. (1) *If $v_i \leq 0$, for all $i = 1, \ldots, p$, and there exist j with $v_j = 0$, then Λ_a is the convex cone generated by the columns of M^{-1} corresponding to $v_r = 0$.*

(2) *If $v_j \leq 0$, for all $j \neq i$ and $v_i > 0$, then $\Lambda_a = \{\lambda \geq 0, |\overline{M}\lambda \geq 0\}$, where $\overline{M}$ is a nonnegative inverse matrix such that $\overline{M}^{-1} = M^{-1}V$, with*

$$V = \begin{pmatrix} v_i & & & & & & \\ & \ddots & & & & & \\ & & v_i & & & & \\ -v_1 & \cdots & -v_{r-1} & 1 & -v_{r+1} & \cdots & -v_p \\ & & & & v_i & & \\ & & & & & \ddots & \\ & & & & & & v_i \end{pmatrix}.$$

In case (1), the linear problems that we have to solve to compute τ^*, have less variables than the initial problem. In case (2), as the procedure to compute τ^* is based on the matrix $\overline{M}^{-1}$, and we have explicitly its expression in terms of M^{-1} and V, it is easy to recompute the values of τ_k.

4. Examples

To illustrate the above results, consider the following multiple linear program taken from Hansen et al. (1989).

$$\begin{aligned} \max \quad & Cx, \\ \text{s.t.} \quad & Ax = b, \\ & x \geq 0, \end{aligned}$$

where

$$C = \begin{pmatrix} 0 & 1 & 0 & 8 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} 4 & 9 & 7 & 10 & 1 & 0 \\ 1 & 1 & 3 & 40 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 6000 \\ 4000 \end{pmatrix}.$$

Considering $\lambda^0 = (1, 1, 3)$ and solving the weighted sum problem, an efficient basic solution of the problem is obtained. The optimal basis is $B = \{x_1, x_4\}$, $K = \{2, 3, 5, 6\}$ and the reduced cost matrix W associated to this basic efficient solution is given by

$$W = \begin{pmatrix} 38/30 & -8/30 & 8/150 & -16/75 \\ 32/30 & 28/30 & 2/150 & -4/75 \\ -39/30 & -21/30 & -59/150 & 3/75 \end{pmatrix}.$$

With no a priori information delimiting the range of values of the objective function weights, only $\lambda_i \geq 0$, we obtain

$$\tau_2 = 47/187, \quad \tau_3 = 43/99, \quad \tau_5 = 167/177,$$

$$\tau_6 = 11/29.$$

Thus, the maximum tolerance percentage is $\tau^* = \tau_2 = 47/187 \approx 25.13\%$. This means that any variation of up to 25.13% in the estimated values of the weights does not change the efficient solution obtained.

4.1. Example 1

Suppose that the DM is able to order the importance of its objectives, and he establishes the following order on the weighting coefficients: $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3$

The procedure to obtain the new maximum tolerance percentage is as follows:

The information can be written as $M\lambda \geq 0$, where

$$M = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Now it follows from algorithm 1, step 1:

$$(W)^t M^{-1} = \begin{pmatrix} 31/30 & -7/30 & -39/30 \\ -1/30 & 7/30 & -21/30 \\ -49/150 & -57/150 & -59/150 \\ -17/75 & -1/75 & 3/75 \end{pmatrix},$$

$w^{5t}(M^{-1})^r \leq 0, \quad \forall r \Rightarrow \tau_5 = +\infty,$

$J = \{2,3,6\}, \quad C = \{\emptyset\}, \quad \tau = +\infty.$

And from step 2:

$t_2 = \min_{k \in J}\{t_k\} = 47/187.$

$$\lambda^2 = \begin{pmatrix} 1 + t_2 \\ 1 + t_2 \\ 3(1 - t_2) \end{pmatrix}, \quad M\lambda^2 \geq 0,$$

and thus

$\tau_2 = t_2, \quad C = \{2\}, \quad \tau^* = \tau_2 \simeq 25.13\%.$

The maximum tolerance percentage has not changed.

4.2. Example 2

If the DM is also able to establish the following lower bounds on the differences between weights:

$0 \leq \lambda_1, \quad 0 \leq \lambda_2 - \lambda_1 \leq 1, \quad 2 \leq \lambda_3 - \lambda_2,$

we have,

$$\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \leq \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \lambda \leq \begin{pmatrix} \infty \\ 1 \\ \infty \end{pmatrix},$$

$$(W)^t M^{-1} = \begin{pmatrix} 31/30 & -7/30 & -39/30 \\ -1/30 & 7/30 & -21/30 \\ -49/150 & -57/150 & -59/150 \\ -17/75 & -1/75 & 3/75 \end{pmatrix}.$$

Using Theorem 1 and step 1. $\tau_2 < +\infty$, $\tau_3 = +\infty$, $\tau_5 = +\infty$, $\tau_6 < +\infty$.

$J = \{2,6\}, \quad C = \{\emptyset\}, \quad \tau = +\infty.$

Using step 2:

$t_2 = \min_{k \in J}\{t_k\} = 47/187.$

$$\lambda^2 = \begin{pmatrix} 1 + t_2 \\ 1 + t_2 \\ 3(1 - t_2) \end{pmatrix}, \quad \lambda^2 \notin \Lambda.$$

Using step 3, $\tau_2 = 1{,}5161290$, $C = \{2\}$.

$\tau_2 < \tau \Rightarrow \tau = \tau_2, \quad J = \{6\}.$

Using step 4, $J \neq \{\emptyset\}$, and again using step 2:

$t_6 = \min_{k \in J}\{t_k\} = 11/29.$

$$\lambda^6 = \begin{pmatrix} 1 - t_6 \\ 1 - t_6 \\ 3(1 + t_6) \end{pmatrix}, \quad \lambda^6 \in \Lambda,$$

and thus $\tau_6 = t_6$, $C = \{2, 6\}$, $\tau^* = \tau_6 = 11/29 \simeq 37.931\%$.

The maximum tolerance percentage has increased to 37.931%.

4.3. Example 3

Suppose that the DM is able to establish the following relations on the weighting coefficients: $0 \leq \lambda_1 \leq \lambda_2$, $3\lambda_2 \leq \lambda_3$.

The information can be written as $M\lambda \geq 0$, where

$$M = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 3 & 1 \end{pmatrix}.$$

Using step 1, $\tau_2 = +\infty$, $\tau_3 = +\infty$, $\tau_5 = +\infty$, $\tau_6 < +\infty$.

$J = \{6\}, \quad C = \{\emptyset\}, \quad \tau = +\infty.$

Using step 2:

$t_6 = \min_{k \in J}\{t_k\} = 11/29,$

$$\lambda^6 = \begin{pmatrix} 1 - t_6 \\ 1 - t_6 \\ 3(1 + t_6) \end{pmatrix}, \quad \lambda^6 \in \Lambda,$$

and thus $\tau_6 = t_6$, $\tau^* = \tau_6 = 11/29 \simeq 37.931\%$.

5. Conclusions and extensions

In a previous paper, Hansen et al. (1989) dealt with the problem of determining the maximum tolerance percentage on weights deviations for a Multiobjective Linear Problem. They developed general results, obtaining useful expressions for the case when there is no information about the weights, and for the case where weights are known to vary in intervals.

In this paper, we enlarge the range of meaningful regions that can be handled easily by this approach, and obtain easy procedures to compute the maximum tolerance percentage to simultaneous and independent variations on the weighting coefficients that generate an efficient solution. The results we develop include as particular cases those of Hansen et al., 1989, and are useful to address the problem when the information that the decision maker is able to give consists of relationships between the importance of the different objectives, in terms of marginal substitution rates.

The procedures that we propose can be extended to any region of weights that are convex cones whose generators are known.

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